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Non-Gaussian diffusion

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Abstract. Diffusion in random-layered media is considered. The non-Gaussian diffusion kinetics is constructed for two models of random-layered structures. The transition of diffusion in such media to asymptotic Gaussian behaviour is studied. A new approach to the description of the kinetics of relativistic electron dechannelling in a crystal is proposed which is based on the concept of diffusion in a random-layered medium developed here.

1. Introduction

Recently, increasing interest has been shown in problems where diffusion is attended by another random process. Since diffusion is a random process, this is a case of 'double randomness' whose physics is basically as follows. In real systems, non-equilibrium perturbations are of a random nature, which causes the structure of the medium to be stochastic. When the relaxation of non-equilibrium perturbations is represented by diffusion equations, one may describe diffusion flows with space- and time-random diffusion coefficients. Of particular interest, as a rule, are flows averaged over scales exceeding the characteristic correlation scales of the diffusion coefficient. Correct procedure of averaging shows that the kinetics of such types of diffusion is given by equations differing from the traditional diffusion equations. The difference can consist, for example, of an additional term containing fourth-order space derivatives and in modification of the diffusion coefficient entering into the term which is square with respect to space derivatives (Dubinko *et al* 1986).

Along with the above situation, where initiation of diffusion and formation of a stochastic structure of the medium are interrelated, there is a set of problems on diffusion in a medium with specified stochastic characteristics, e.g., the problem of wave propagation in random inhomogeneous media (Klyatskin 1980). Among these is the problem studied in this paper, namely diffusion in a random-layered medium. Two exactly solvable models are used as an example to construct the non-Gaussian diffusion kinetics in random-layered media.

In § 2 we formulate the problem of a distribution function describing diffusion in a random-layered medium. A random structure is modelled by a telegraph process. It is shown that the kinetics is non-local. The role of the higher-order moments of the probability distribution describing the structure of the medium is discussed. The criterion of the Gaussian behaviour of diffusion is presented.

In § 3, the random-layered structure is modelled by a Poisson process. It is shown that the problem of finding the stochastic characteristics of diffusion in a Poisson-layered medium may be formulated as a problem of construction of thermodynamics

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of a one-dimensional gas with a positive pairwise interaction potential. The diffusion kinetics in such a medium in the limit of a 'non-interacting particles gas' is described by an equation similar to the linear Boltzmann equation. The range of applicability of the Gaussian approximation for description of diffusion in a Poisson random-layered medium is studied.

Section 4 proposes a new approach to construction of the dechannelling kinetics of relativistic electrons in crystals based on the concept of diffusion in random-layered media developed in this paper.

2. Diffusion in a random structure modelled by a telegraph process

It is well known that the stochastic equation

$$\dot{x}(\tau) = \eta(\tau) \tag{2.1}$$

where $\eta(\tau)$ is the white noise, leads to the diffusion equation for the distribution function $f(x-x_0;\ \tau)$ of the quantity x†:

$$f(x-x_0;\ \tau) = \langle \delta(x - x_\eta(\tau;\ x_0)) \rangle_\eta \tag{2.2}$$

$$\frac{\partial f(x-x_0;\ \tau)}{\partial \tau} = D\frac{\partial^2}{\partial x^2} f(x-x_0;\ \tau) \tag{2.3}$$

$$f(x-x_0;\ \tau=0) = \delta(x-x_0)$$

where $x_\eta(\tau;\ x_0)$ is the formal solution of (2.1) depending on the initial data x_0, $\langle \ldots \rangle_\eta$ means averaging over the white noise performed according to the Wick theorem with a relation determined by the pair correlator $\langle \eta(\tau)\eta(\tau) \rangle_\eta = 2D\delta(\tau - \tau')$ and D is the diffusion coefficient.

Consider now the situation of the diffusion process being switched on and off in a random way. For example, in the problem of multiple scattering (Ter-Mikaelian 1972), we deal with the transit of a fast charged particle through random-thickness plates of matter separated by random-width vacuum gaps. In the plane perpendicular to the particle transit direction through the stochastic structure there is diffusion in every plate and free motion in the vacuum gaps. The alternate random regions, i.e. diffusion and no-diffusion time intervals, are here identified as the random-layered medium. Accordingly, we shall modulate the white noise by another process $\zeta(\tau)$ statistically independent of the white noise; that is, let us consider the following stochastic equation:

$$\dot{x}(\tau) = \eta(\tau)\zeta(\tau).$$

For certain specified realisations of the processes $\eta(\tau)$ and $\zeta(\tau)$ this equation has the formal solution

$$x(\tau) = x_0 + \int_0^\tau \mathrm{d}t\, \eta(t)\zeta(t) \qquad x_0 \equiv x(\tau)\Big|_{\tau=0}.$$

† We consider the one-dimensional case, though all the arguments are easy to generalise to the three-dimensional case.

The simultaneous statistical characteristics of the process $x(\tau)$ can be found if the distribution function $f(x-x_0;\tau)$ defined in (2.2) is known:

$$f(x-x_0;\tau)=\left\langle \delta\left(x-x_0-\int_0^\tau \mathrm{d}t\,\eta(t)\zeta(t)\right)\right\rangle_{\eta,\zeta} \tag{2.4}$$

$$f(x-x_0;\tau=0)=\delta(x-x_0)$$

where $\langle\ldots\rangle_{\eta,\zeta}$ means averaging over the white noise and the realisations of the process $\zeta(\tau)$, where the order of averaging can be changed due to the statistical independence of the processes. The procedure of averaging over the random process $\zeta(\tau)$ will be defined later, after the process $\zeta(\tau)$ has been defined. Let us now average (2.4) over $\eta(\tau)$. By representing expression (2.4) as a Fourier integral

$$f(x-x_0;\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k\,\mathrm{e}^{\mathrm{i}k(x-x_0)}\bar{f}(k,\tau) \tag{2.5}$$

we find the Fourier image as follows:

$$\bar{f}(k,\tau)=\left\langle \exp\left(-k^2D\int_0^\tau \mathrm{d}t\,\zeta^2(t)\right)\right\rangle_\zeta. \tag{2.6}$$

We shall consider in this section the process $\zeta(\tau)$ as some version of the telegraph process. Its typical realisation represented in figure 1 is a step function equal to 0 or 1, the 'jump' from 0 to 1 and back occurring at random points τ_i. The jumps of the process $\zeta(\tau)$ provide switching on and off the diffusion mechanism. The process $\zeta(\tau)$ is defined as follows:

$$\zeta(\tau)=\tfrac{1}{2}(1+z(\tau)) \tag{2.7}$$

where the random process $z(\tau)$ is given by the following relations (Klyatskin 1980, Kats 1967):

$$z(\tau)=(-1)^{n(0,\tau)} \qquad z(0)=1 \qquad z^2(\tau)\equiv 1 \tag{2.8}$$

where $n(0,\tau)$ is the number of points τ_i falling within the interval $(0,\tau)$ which is a random quantity with the Poisson distribution law:

$$\mathrm{Prob}(n(\tau_1,\tau_2)=n)=\frac{(\bar{n}(\tau_1,\tau_2))^n}{n!}\exp(-\bar{n}(\tau_1,\tau_2))$$

$$\bar{n}(\tau_1,\tau_2)=\nu|\tau_1-\tau_2|$$

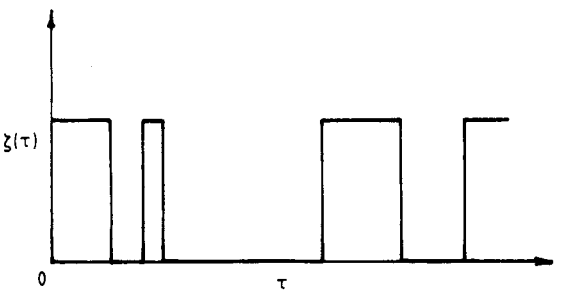

Figure 1. One of the possible realisations of the random process $\zeta(\tau)$.

where ν is the average number of points per unit time. The first two moments of the process $z(\tau)$ are:

$$\langle z(\tau)\rangle_z = e^{-2\nu\tau} \qquad \langle z(\tau_1)z(\tau_2)\rangle_z = e^{-2\nu(\tau_1-\tau_2)} \qquad (\tau_1 \geqslant \tau_2)$$

and the higher-order moments satisfy the recurrence relation

$$\langle z(\tau_1)\ldots z(\tau_n)\rangle_z = \langle z(\tau_1)z(\tau_2)\rangle_z \langle z(\tau_3)\ldots z(\tau_n)\rangle_z \qquad (\tau_1 \geqslant \tau_2 \geqslant \ldots \geqslant \tau_n). \tag{2.9}$$

These expressions define the procedure of averaging over realisations of the process $z(\tau)$ denoted by $\langle\ldots\rangle_z$.

In view of relations (2.7) and (2.8), we represent (2.6) as

$$\bar{f}(k,\tau) = \exp\left(-\frac{k^2 D\tau}{2}\right)\left\langle \exp\left(-\frac{k^2 D}{2}\int_0^\tau dt\, z(t)\right)\right\rangle_z.$$

By averaging over z and using the definition (2.5), we obtain

$$f(x-x_0;\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(x-x_0)} \exp\left[-\left(\frac{k^2 D}{2}+\nu\right)\tau\right] \times\left[\cosh\left(\nu^2+\frac{k^4D^2}{4}\right)^{1/2}\tau + \nu\left(\nu^2+\frac{k^4D^2}{4}\right)^{-1/2}\sinh\left(\nu^2+\frac{k^4D^2}{4}\right)^{1/2}\tau\right]$$

$$f(x-x_0;0) = \delta(x-x_0) \qquad \frac{\partial}{\partial\tau}f(x-x_0;\tau)\bigg|_{\tau=0} = \frac{D}{2}\frac{\partial^2}{\partial x^2}\delta(x-x_0).$$

The distribution function constructed in this way satisfies the following kinetic equation:

$$\frac{\partial f(x-x_0;\tau)}{\partial\tau} = \frac{D}{2}\frac{\partial^2}{\partial x^2}f(x-x_0;\tau) + \frac{D^2}{4}\frac{\partial^4}{\partial x^4}\int_0^\tau dt\, e^{-2\nu(\tau-t)} \times\int_{-\infty}^{\infty} dx' G(x-x';\tau-t) f(x'-x_0;t) \tag{2.10}$$

where

$$G(x,\tau) = (2\pi D\tau)^{-1/2}\exp(-x^2/2D\tau).$$

As is suggested by this equation, the diffusion in a random-layered medium is non-local. The first term on the right-hand side of (2.10) is a standard diffusion term due to the first moment of the probability distribution describing the medium structure while the integral term in (2.10) is due to higher-order moments. Here we consider the situation when higher-order moments should be taken into account in addition to the first one, as, e.g., in the problem of intermittency in random media (Zel'dovich *et al* 1987).

For large ν, when the medium structure looks like a thick comb, (2.10) becomes local:

$$\frac{\partial f(x-x_0;\tau)}{\partial\tau} = \frac{D}{2}\frac{\partial^2}{\partial x^2}\left(1+\frac{D}{4\nu}\frac{\partial^2}{\partial x^2}\right)f(x-x_0;\tau). \tag{2.11}$$

Let us show that this equation has no unstable (increasing with τ) solution. Using relation (2.5), obtain

$$\frac{\partial\bar{f}(k,\tau)}{\partial\tau} = -\frac{k^2 D}{2}\left(1-\frac{k^2 D}{4\nu}\right)\bar{f}(k,\tau).$$

Instability might be the case for $1-k^2D/4\nu<0$, i.e. for $k^2>4\nu/D$, which corresponds to the scale $x<(D/4\nu)^{1/2}$. This inequality means that the kinetics is realised within a single switching of the diffusion mechanism, i.e. we disregard the statistics of random lamination which was essentially taken into account in deriving (2.10) and (2.11). Thus, the applicability conditions of (2.10) and therefore of (2.11) are violated. Hence, (2.11) has no unstable solution on the scales of its applicability.

Obviously, when $\tau \gg \nu^{-1}$, we deal with Gaussian diffusion and (2.10) transforms into the equation

$$\frac{\partial f(x-x_0;\,\tau)}{\partial \tau}=\frac{D}{2}\frac{\partial^2}{\partial x^2}f(x-x_0;\,\tau).$$

Since diffusion and no-diffusion intervals have the same statistics, the averaged diffusion coefficient is in this equation half as large as that for a homogeneous medium (see (2.3)).

Thus, diffusion in a telegraph-type medium is given by the integro-differential kinetic equation (2.10). This is equivalent to the second-order differential equation

$$\frac{\partial^2 f(x-x_0;\,\tau)}{\partial \tau^2}+\left(2\nu-D\frac{\partial^2}{\partial x^2}\right)\frac{\partial f(x-x_0;\,\tau)}{\partial \tau}-\nu D\frac{\partial^2 f(x-x_0;\,\tau)}{\partial x^2}=0$$

$$f(x-x_0;\,0)=\delta(x-x_0) \qquad \frac{\partial}{\partial \tau}f(x-x_0;\,\tau)\bigg|_{\tau=0}=\frac{D}{2}\frac{\partial^2}{\partial x^2}\delta(x-x_0).$$

This fact implies that the medium stochastic structure is described by a process with two possible states. Due to statistics of the medium the integral term on the right-hand side of (2.10) contains only a pair correlator, as is the case for a random medium with Gaussian properties (Dubinko *et al* 1986). This is evidently caused by stochastic property (2.9) of a telegraph random-layered medium.

3. Poisson random-layered structure

Remembering the example of the preceding part of the problem on multiple scattering, let us consider a random alternation of plates of matter which is given by the Poisson (pulsed) process $\zeta(\tau)=\Sigma_i g(\tau-\tau_i)$, where the points τ_i are uniformly distributed in the interval $(0,\tau)$ and their number n obeys the Poisson law with the parameter $\bar{n}=\nu\tau$. Restricting ourselves to the case of plane-parallel plates of thickness Δ, we choose the function $g(\tau)$ describing the pulse shape as

$$g(\tau)=\chi(\tau)\chi(\Delta-\tau) \tag{3.1}$$

where $\chi(\tau)$ is the Heaviside step and

$$\chi(\tau)=\begin{cases}1 & \tau\geqslant 0\\ 0 & \tau<0.\end{cases}$$

Here the plate thickness Δ and parameter ν (ν^{-1} is the average plate distance) must satisfy the condition $\Delta\ll\nu^{-1}$, provided there is no penetration of matter from one plate into another. Note that the probability of interpenetration of plates or superposition of the pulses $g(\tau)$, i.e. of an event for which the distance between neighbouring points τ_i is smaller than Δ, is $p=1-e^{-\Delta\nu}\simeq\Delta\nu$.

The averaging over realisation of the Poisson process $\zeta(\tau)$ denoted by $\langle \ldots \rangle_\zeta$ is defined as follows:

$$\langle \ldots \rangle_\zeta = e^{-\nu\tau} \sum_{n=0}^{\infty} \frac{\nu^n}{n!} \int_0^\tau d\tau_1 \ldots \int_0^\tau d\tau_n \ldots . \tag{3.2}$$

By averaging expression (2.6) and taking account of the explicit form of the function $g(\tau)$ defined by (3.1), we obtain

$$\bar{f}(k, \tau) = e^{-\nu\tau} \sum_{n=0}^{\infty} \frac{\nu^n}{n!} \int_0^\tau d\tau_1 \ldots \int_0^\tau d\tau_n \exp\left(-k^2 D \sum_{i,j=1}^{n} \phi(|\tau_i - \tau_j|)\right) \tag{3.3}$$

where

$$\phi(|\tau_i - \tau_j|) = \int_0^\tau d\tau' g(\tau' - \tau_i) g(\tau' - \tau_j) = (\Delta - |\tau_i - \tau_j|)\chi(\Delta - |\tau_i - \tau_j|). \tag{3.4}$$

We represent expression (3.3) as

$$\bar{f}(k, \tau) = e^{-\nu\tau} \sum_{n=0}^{\infty} z^n Q_n$$

where the parameter $z = \nu e^{-k^2 D\Delta}$ will be identified in terms of the equilibrium statistical mechanics as 'activity', and Q_n

$$Q_n = \frac{1}{n!} \int_0^\tau d\tau_1 \ldots \int_0^\tau d\tau_n \exp\left(-2k^2 D \sum_{i>j=1}^{n} \phi(|\tau_i - \tau_j|)\right)$$

will be called the 'configuration integral'. By expanding the configuration integral in terms of connected groups, the following expression for $\bar{f}(k, \tau)$ is easily obtained:

$$\bar{f}(k, \tau) = e^{-\nu\tau} \exp\left(\tau \sum_{l=1}^{\infty} b_l z^l\right) \tag{3.5}$$

where b_l are the group integrals

$$b_l = \frac{1}{\tau l!} \int_0^\tau d\tau_1 \ldots \int_0^\tau d\tau_l \sum \prod f_{ij}$$

f_{ij} being the Mayer functions, $f_{ij} = \exp[-2k^2 D\phi(|\tau_i - \tau_j|)] - 1$. The expression for b_l includes the sum of products of the functions f_{ij} for particles bound by 'f-bonds' (Isihara 1971).

By analogy with statistical mechanics, the quantity $2k^2 D$ can be called the 'reciprocal temperature'. Thus, calculation of the Fourier transform $\bar{f}(k, \tau)$ (see (3.3)) of the distribution function describing the diffusion kinetics in the Poisson random-layered structure formally reduces to calculation of a large statistical sum of a one-dimensional gas of particles in the 'volume' τ with a positive potential of a pairwise interaction $\phi(\tau) \geqslant 0$. It is easy to show that expansion in powers of 'activity' is in fact a series in powers of the parameter $\Delta\nu$, which characterises the degree of imperfection of this one-dimensional gas.

We shall construct the diffusion kinetics in the Poisson random-layered medium in the 'ideal gas approximation'. The reason is that the layered structure under consideration consists of plates of an equal thickness Δ separated by random gaps distributed by an exponential law with the parameter ν, and in view of the condition

$\Delta\nu \ll 1$ can be treated by analogy with statistical mechanics as a one-dimensional ideal gas. In other words only the first term with $l = 1$ should be retained in the exponent of formula (3.5). Noting that

$$b_1 = \frac{1}{\tau}\int_0^\tau d\tau_1 = 1$$

and $z = \nu\, e^{-k^2 D\Delta}$, we obtain $\bar{f}(k, \tau) = \exp[\nu\tau(e^{-k^2 D\Delta} - 1)]$ and hence:

$$f(x - x_0; \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(x-x_0)} \exp[\nu\tau(e^{-k^2 D\Delta} - 1)]$$

$$= e^{-\nu\tau}\delta(x - x_0) + e^{-\nu\tau}\sum_{m=1}^{\infty}\frac{(\nu\tau)^m}{m!}\frac{1}{\sqrt{4\pi D\Delta m}}\exp\left(-\frac{(x-x_0)^2}{4D\Delta m}\right). \tag{3.6}$$

This distribution is non-Gaussian and satisfies a kinetic equation of the linear Boltzmann equation type:

$$\frac{\partial f(x - x_0; \tau)}{\partial\tau} = \frac{\nu}{\sqrt{4\pi D\Delta}}\int_{-\infty}^{\infty} dx' \exp\left(-\frac{(x'-x)^2}{4D\Delta}\right)(f(x' - x_0; \tau) - f(x - x_0; \tau)) \tag{3.7}$$

$$f(x - x_0; \tau = 0) = \delta(x - x_0).$$

The 'double randomness' mentioned in § 1 manifests itself here as the Boltzmann structure of the kinetic equation (3.7), which is due to the Poisson process, and in the diffusive character of the kernel of this integral equation. Asymptotically, at large times, $\nu\tau \gg 1$ (after transit through a large number of plates) the diffusion becomes Gaussian and, as is easy to see, the kinetic equation (3.7) transforms into the traditional diffusion equation

$$\frac{\partial f(x - x_0; \tau)}{\partial\tau} = D\Delta\nu\frac{\partial^2}{\partial x^2} f(x - x_0; \tau) \tag{3.8}$$

$$f(x - x_0; \tau = 0) = \delta(x - x_0)$$

with an averaged diffusion coefficient which is $\Delta\nu$ times smaller than that in a homogeneous medium (cf (2.3)).

Now we introduce the new variables $y = x/\sqrt{4D\Delta}$, $y_0 = x_0/\sqrt{4D\Delta}$ and $\beta = \nu\tau$, in order to carry out a quantitative analysis of diffusion characteristics in a Poisson random-layered medium. In these variables the solution of (3.7) denoted by $f_1(y - y_0; \beta)$ is

$$f_1(y - y_0; \beta) = e^{-\beta}\delta(y - y_0) + e^{-\beta}\sum_{m=1}^{\infty}\frac{\beta^m}{m!}\frac{1}{\sqrt{\pi m}}\exp\left(-\frac{(y-y_0)^2}{m}\right)$$

$$\int_{-\infty}^{\infty} dy\, f_1(y - y_0; \beta) = 1. \tag{3.9}$$

The solution of (3.8), which we denote by $f_2(y - y_0; \beta)$ in the same variables can be

represented in the form

$$f_2(y-y_0;\beta)=\frac{1}{\sqrt{\pi\beta}}\exp\left(-\frac{(y-y_0)^2}{\beta}\right)$$

$$\int_{-\infty}^{\infty}\mathrm{d}y f_2(y-y_0;\beta)=1. \tag{3.10}$$

Consider, using the functions $f_1(y-y_0;\beta)$ and $f_2(y-y_0;\beta)$, the evolution of an initial distribution in a Poisson random-layered medium and in a homogeneous amorphous medium characterised by the averaged diffusion coefficient $D\Delta\nu$. Choosing the

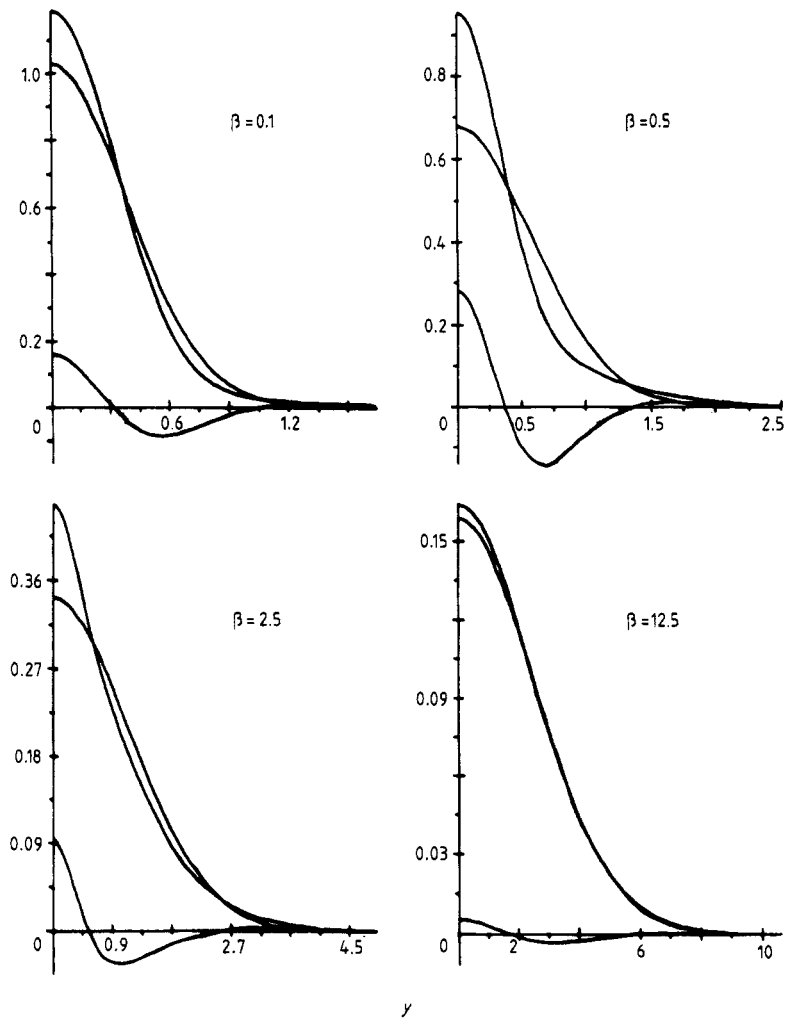

Figure 2. The vertical axis is the range of the function: $\varphi_1(y;\alpha,\beta)$ is the curve assuming maximum values at $y=0$ for all values of β; $\varphi_2(y;\alpha,\beta)$ is the curve assuming an intermediate value at $y=0$; and $(\varphi_1(y;\alpha,\beta)-\varphi_2(y;\alpha,\beta))$ is the curve assuming the minimum values at $y=0$ for all values of β. The numerical value of the parameter α is 0.2. The parameter β is interpreted as time. Note that both the horizontal scale (y values) and the vertical scale (values of $\varphi_1, \varphi_2, \varphi_1-\varphi_2$) vary from one β value to another.

Gaussian distribution with the variance α, $\varphi(y_0, \alpha) = (\pi\alpha)^{-1/2} \exp(-y_0^2/\alpha)$ to be the initial condition, we find

$$\varphi_1(y; \alpha, \beta) = \int_{-\infty}^{\infty} dy_0 \, \varphi(y_0, \alpha) f_1(y - y_0; \beta)$$

$$= e^{-\beta} \sum_{m=0}^{\infty} \frac{\beta^m}{m!} \frac{1}{\sqrt{\pi(\alpha + m)}} \exp\left(-\frac{y^2}{\alpha + m} \right) \tag{3.11}$$

$$\varphi_2(y; \alpha, \beta) = \int_{-\infty}^{\infty} dy_0 \, \varphi(y_0, \alpha) f_2(y - y_0; \beta) = \frac{1}{\sqrt{\pi(\alpha + \beta)}} \exp\left(-\frac{y^2}{\alpha + \beta} \right). \tag{3.12}$$

Figure 2 shows the functions $\varphi_1(y; \alpha, \beta)$ and $\varphi_2(y; \alpha, \beta)$ for $y \geq 0$ at fixed 'moments of time' β. As is seen, for small β the diffusion in a Poisson random-layered medium is essentially non-Gaussian, whereas for large β there appears an asymptotic Gaussian behaviour characterised by the averaged diffusion coefficient $D\Delta\nu$.

To conclude this section, we note that diffusion in random-layered media is of a more complex statistical character than diffusion in its traditional sense. The form of the kinetic equations describing the diffusion kinetics in such media depends on the stochastic properties of the medium. Thus, in a medium approximated by the Poisson process, diffusion is given by the kinetic equation (3.7) of Boltzmann type, while in a medium modelled by a telegraph process the diffusion kinetics is given by (2.10). The diffusion processes given by (2.10) and (3.7) are non-Gaussian. Asymptotically, $\nu\tau \gg 1$, diffusion in random-layered media becomes Gaussian and is characterised by an averaged diffusion coefficient which is smaller than in the case of a homogeneous medium.

4. Channelling, dynamic stochastisation and diffusion

In this section we shall show how, on the basis of the concept of diffusion in a Poisson random-layered medium formulated in the preceding section, a new approach to the description of the kinetics of channelled relativistic electrons in crystals can be developed.

When relativistic electrons with the energy ε are incident onto a crystal along one of the crystallographic axes (the z axis), they can experience a channelling effect, i.e. move in channels formed by chains of crystal atoms. That is to say, motion of an electron is determined chiefly by the continuous potential of crystal atom chains, namely the crystal potential averaged over the z coordinate (Ahiezer and Shul'ga 1987)

$$U(\boldsymbol{\rho}) = \frac{1}{T} \int dz \sum_k u(\boldsymbol{r} - \boldsymbol{r}_k)$$

where $u(\boldsymbol{r} - \boldsymbol{r}_k)$ is the potential energy of the interaction of an electron with a lattice atom situated at the point $\boldsymbol{r}_k$, $\boldsymbol{\rho} = (x, y)$ are the coordinates in the plane of the orthogonal z axis, and T is the crystal thickness.

The transverse motion energy $\varepsilon_\perp$ of an electron is defined as $\varepsilon_\perp = \frac{1}{2}\varepsilon\dot{\boldsymbol{\rho}}^2 + U(\boldsymbol{\rho})$, $|\varepsilon_\perp| \ll \varepsilon$. Because $\varepsilon_\perp$ changes as a result of electron scattering by thermal vibrations of lattice atoms, the finite motion of the electron (channelling) can become infinite (above-barrier motion) with respect to the z axis. This effect is called dechannelling. It has usually been assumed that particle motion in a channel is periodic and the

dechannelling process develops adiabatically with the depth of particle penetration into the crystal. Accordingly, the fast electron kinetics in a channel is described by Fokker-Planck equations, with the diffusion coefficients averaged over the particle motion period. Recently, however, two new effects have been revealed in the problem of relativistic charged particle motion through crystals. One of them is associated with the fact that in the field $U(\boldsymbol{\rho})$ (one of its possible realisations is shown in figure 3) the dynamics can be not only regular, but also stochastic. This phenomenon was discovered in numerical experiments aimed at the construction of Poincaré mappings for a dynamic trajectory in the field $U(\boldsymbol{\rho})$. The stochasticity region in the Poincaré mapping depends both on the initial conditions determining the channelled electron trajectory and on $\varepsilon_\perp$. For $\varepsilon_\perp = 0.5U_c$, where $U_c = -24$ eV is the potential energy at the saddle point (see figure 3), as is found by an analysis of the Poincaré mapping, practically all trajectories in the channel are stochastic (Ahiezer and Shul'ga 1987, Shul'ga *et al* 1987). The other effect is that in a wide particle energy range, when channelled electrons pass through the region of crystal thermal vibrations, there are strong fluctuations of the transverse energy integral $\langle \delta\varepsilon_\perp^2 \rangle^{1/2} \sim \varepsilon_\perp$ (Bazylev *et al* 1986, Shul'ga 1986). Thus, there arises the question of how these factors are to be allowed for and how they influence the dechannelling kinetics.

The energy of the electron transverse motion changes mainly† when electrons pass through the region of the order of the atom thermal vibration amplitude $\langle w^2 \rangle^{1/2}$, and if distances from the channel axis are over $\langle w^2 \rangle^{1/2}$, it remains constant. In the latter case the electron motion, as has been mentioned, is determined by the continuous

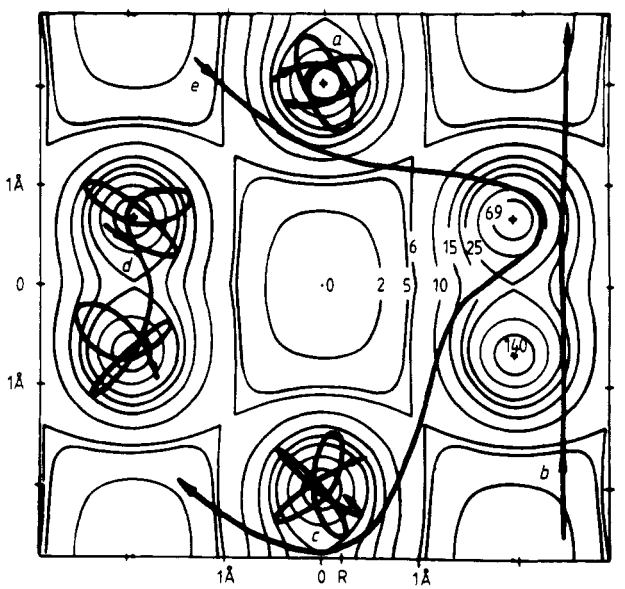

Figure 3. Equipotential surfaces of the continuous potential energy of the interaction of a silicon crystal with an electron moving along the ⟨110⟩ axis and typical regular (*a*, *b*) and chaotic (*c*, *d*, *e*) trajectories of channelled and above-barrier electrons in the plane perpendicular to the axis ⟨110⟩. The numbers on the lines indicate negative values of $U(\boldsymbol{\rho})$ in eV; $U(\boldsymbol{\rho})$ is assumed to be zero at the centre of a cell. Figure taken from Shul'ga (1986).

† Multiple scattering of a moving particle by the crystal electron subsystem will be neglected.

potential of the atom chain. Thus, in the case of electron channelling, the transverse motion energy will only change at short trajectory sections (the channel being much wider than $\langle w^2\rangle^{1/2}$), crossing the thermal vibration region. For chaotic motion of an electron (see, e.g., trajectory d in figure 3) time intervals between successive transits through the thermal vibration region are random. Therefore, electron channelling in dynamic chaos is a problem of multiple scattering of a fast charged particle in a random-layered medium (see figure 4). The 'double randomness' discussed in § 1 seems to manifest itself here as randomness of instants of electron inflight into 'plates', i.e. regions of crystal thermal vibrations, caused by dynamic stochasticity. It also appears in the diffuse character of motion in every 'plate', i.e. during motion in the themal vibration region. We emphasise that both the random processes are realised in the plane perpendicular to the z axis.

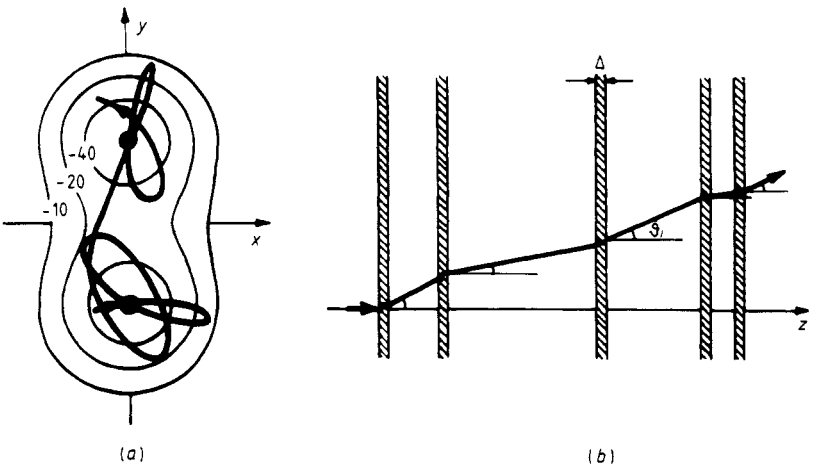

Figure 4. (a) Stochastic trajectory of an electron in a channel in the plane perpendicular to the ⟨110⟩ axis (z axis) in a Si crystal. The shaded regions are the regions of crystal lattice atom thermal vibrations. (b) 'Development' of the stochastic trajectory of penetration into the crystal (z is the penetration depth). ϑ_i ($i = x, y$) are components of the two-dimensional vector, the angle between the z axis and the electron momentum.

Let us study the dechannelling kinetics for the case of dynamic chaos and large fluctuations of the transverse motion energy integral using as an example the model constructed in the preceding section. The multiple scattering in a random-layered medium will be characterised by the two-dimensional vector $\boldsymbol{\vartheta}$ (see figure 4(b)). The kinetic equation describing the evolution of the distribution function $f(\boldsymbol{\vartheta}, z)$ of electrons in the angles $\boldsymbol{\vartheta}$ at the depth of penetration into a crystal, z, is as follows, according to (3.7):

$$\frac{\partial f(\boldsymbol{\vartheta}, z)}{\partial z} = \frac{\nu}{4\pi D\Delta}\int \mathrm{d}\boldsymbol{\vartheta}' \exp\left(-\frac{(\boldsymbol{\vartheta}'-\boldsymbol{\vartheta})^2}{4D\Delta}\right)(f(\boldsymbol{\vartheta}', z)-f(\boldsymbol{\vartheta}, z)) \qquad (4.1)$$

$$f(\boldsymbol{\vartheta}, z=0) = \delta(\boldsymbol{\vartheta})$$

where ν^{-1} is the average plate separation†, Δ the plate thickness and D the diffusion coefficient in the atom vibration region.

† The velocity of light is assumed equal to unity, $c = 1$; thus, the z component of the electron velocity is $v_z \simeq c = 1$.

For $\nu z \gg 1$, corresponding to the case of electron transit through a larger number of plates, the diffusion becomes Gaussian, and (4.1) transforms into the diffusion equation

$$\frac{\partial f(\vartheta, z)}{\partial z} = D\Delta\nu \frac{\partial^2}{\partial \vartheta^2} f(\vartheta, z) \tag{4.2}$$

with an averaged diffusion coefficient.

Thus, the above proposed approach to the description of the dechannelling process as diffusion of a particle in a random-layered medium leads to the following conclusions. In the case of a single transit of an electron through the lattice atom thermal vibration region, fluctuations of the transverse motion energy integral $\langle \delta\varepsilon_\perp^2 \rangle^{1/2}$ can be higher than $|\varepsilon_\perp|$, so the electron leaves the channel very soon (Bazylev *et al* 1986, Shul'ga 1986). The dechannelling process in this case occurs at the depth $z \sim \nu^{-1}$, where the particle distribution in angles is non-Gaussian. This is the case, e.g., for channelling of electrons with the energy $\varepsilon \sim 1$ GeV in a silicon crystal along the $\langle 110 \rangle$ axis. As the electron energy increases the quantity $\langle \delta\varepsilon_\perp^2 \rangle^{1/2}$ decreases. If $\langle \delta\varepsilon_\perp^2 \rangle^{1/2} \sim |\varepsilon_\perp|$, then the electron, before leaving the channel, will cross the atom thermal vibration region several times. Dechannelling in this case is not adiabatic, but the process can be studied on the basis of the diffusion equation (4.2) with the diffusion coefficient averaged over realisations of stochastic trajectories of particles in the channel, the said realisations depending both on dynamic stochastisation and on transverse motion energy integral fluctuations.

When $\langle \delta\varepsilon_\perp^2 \rangle^{1/2} \ll |\varepsilon_\perp|$, the dechannelling process develops adiabatically, with increasing penetration depth of the particle. This process can be described on the basis of the standard diffusion equation with the diffusion coefficient averaged over stochastic trajectories generated by the chaotic dynamics of particles in the channel (fluctuations of the transverse motion energy integral may be neglected in this case).

5. Conclusion

We have studied diffusion in random-layered media, using two exactly solvable models as examples. The statistical structure of such diffusion is shown to be of a non-Gaussian character, unlike the traditional concept of diffusion. Based on the results of the study, a new approach is proposed to describe the dechannelling of relativistic electrons in a crystal as diffusion of a particle in a random-layered medium.

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